

Title	石油井戸に関するファジィ微分方程式の最適化問題 (関数方程式と数理モデル)
Author(s)	Saito, Seiji; Ishii, Hiroaki
Citation	数理解析研究所講究録 (2003), 1309: 240-248
Issue Date	2003-02
URL	<a href="http://hdl.handle.net/2433/42891">http://hdl.handle.net/2433/42891</a>
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

# 石油井戸に関するファジィ微分方程式の最適化問題

大阪大学大学院情報科学研究科情報数理学専攻 齋藤誠慈 (Seiji SAITO)

Information Science and Technology, Osaka University

Suita, Osaka, 565-0871

(saito@ist.osaka-u.ac.jp)

大阪大学大学院情報科学研究科情報数理学専攻 石井博昭 (Hiroaki ISHII)

Information Science and Technology, Osaka University

Suita, Osaka, 565-0871

(ishii@ist.osaka-u.ac.jp)

**Abstract** In this study we give a new representation of fuzzy numbers with bounded supports and also we show that a fuzzy number means a bounded continuous curve in the two-dimensional metric space. Our aims of this research are to discuss optimization problems with objective functions and constraints both of which are  $L$ -fuzzy functions and to consider oil well equations which are represented by fuzzy differential equations  $C'_L(t) + D_L C_L(t) = 0$ , where  $t$  is the time,  $0 \in \mathbf{R}$ ,  $C_L(t)$  an  $L$ -fuzzy function and  $D_L$  a constant  $L$ -fuzzy number by applying the above criteria of  $L$ -optimization problems. Moreover we get an extension of the maxi-max theorem of optimization problems with an infinite number of constraints and a objective function which consists of infinite series.

**Keywords:** Fuzzy Numbers; Fuzzy Differential Equations; Attractive Set ; Couple Parametric Representation;

## 1 Set of fuzzy numbers

Let  $I = [0, 1]$ . We define the following set of fuzzy numbers, where a fuzzy number  $x$  is characterized by a membership function  $\mu_x$  as follows (cf. [2, 3]):

**Definition 1** Denote

$\mathcal{F}_b^{st} = \{\mu_x : \mathbf{R} \rightarrow I \text{ satisfying (i) - (iv) below}\}.$

- (i) There exists a unique  $m \in \mathbf{R}$  such that  $\mu_x(m) = 1$ ;
- (ii) The support set  $\text{supp}(\mu_x) = \text{cl}(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$  is bounded in  $\mathbf{R}$ ;
- (iii) Let  $J = \{\xi \in \mathbf{R} : 0 < \mu_x(\xi)\}$ .  $\mu_x$  is strictly quasi-convex on  $J$ , i.e.,  $\mu_x(\lambda\xi_1 + (1-\lambda)\xi_2) >$

$\min[\mu_x(\xi_1), \mu_x(\xi_2)]$  for  $0 < \lambda < 1$  and  $\xi_1, \xi_2 \in J$  such that  $\xi_1 \neq \xi_2$ ;

- (iv)  $\mu_x$  is upper semi-continuous on  $\mathbf{R}$ .

In usual case a fuzzy number  $x$  satisfies quasi-convex on  $\mathbf{R}$ , i.e.,

$$\mu_x(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \min[\mu_x(\xi_1), \mu_x(\xi_2)]$$

for  $0 \leq \lambda \leq 1$  and  $\xi_1, \xi_2 \in \mathbf{R}$ . Condition (iii) plays an important role in proving properties of membership function  $\mu_x$  in Theorem 1, where we show significant properties concerning the end-points of the  $\alpha$ -cut set  $L_\alpha(\mu_x) = \{\xi \in \mathbf{R} : \mu_x(\xi) \geq \alpha\}$ .

In the similar way as [2, 3] we consider the following parametric representation of  $\mu_x \in \mathcal{F}_b^{st}$

such that

$$x_1(\alpha) = \min L_\alpha(\mu_x), \quad x_2(\alpha) = \max L_\alpha(\mu_x)$$

for  $0 < \alpha \leq 1$  and that

$$x_1(0) = \min cl(supp(\mu_x)), x_2(0) = \max cl(supp(\mu_x)).$$

In what follows we denote a fuzzy numbers  $x$  by  $(x_1, x_2)$ , i.e.,  $x = (x_1, x_2)$ .

By applying the above extension principle and the representation of fuzzy numbers we get the following results.

1) Addition. Let  $x = (x_1, x_2)$ ,  $y = (y_1, y_2) \in \mathcal{F}_b^{st}$ . We get the addition

$$\begin{aligned} \mu_{x+y}(\xi) &= \sup_{\xi=\xi_1+\xi_2} \min[\mu_x(\xi_1), \mu_y(\xi_2)] \\ &= \sup\{\alpha \in I : \xi = \xi_1 + \xi_2, \quad \xi_1 \in x_\alpha, \xi_2 \in y_\alpha\} \quad \text{holds;} \\ &= \sup_{\xi \in [x_1(\alpha)+y_1(\alpha), x_2(\alpha)+y_2(\alpha)]} \alpha, \end{aligned}$$

which means that  $x + y = (x_1 + y_1, x_2 + y_2)$ . Here  $x_\alpha = L_\alpha(\mu_x)$  etc.

2) Subtraction. It follows that

$$\mu_{x-y}(\xi) = \sup\{\alpha \in I : \xi = \xi_1 - \xi_2, \quad \xi_1 \in x_\alpha, \xi_2 \in y_\alpha\}$$

means that  $x - y = (x_1 - y_2, x_2 - y_1)$ .

3) Product. It follows that

$$\mu_{xy}(\xi) = \sup\{\alpha \in I : \xi = \xi_1 \xi_2, \quad \xi_1 \in x_\alpha, \xi_2 \in y_\alpha\}$$

means that the following relation.

$$xy = \begin{cases} (x_1 y_1, x_2 y_2) & (0 \leq x_1, 0 \leq y_1) \\ (x_2 y_1, x_2 y_2) & (0 \leq x_1, y_1 \leq 0 \leq y_2) \\ (x_2 y_1, x_1 y_2) & (0 \leq x_1, y_2 \leq 0) \\ (x_1 y_2, x_2 y_2) & (x_1 \leq 0 \leq x_2, 0 \leq y_1) \\ (\min\{x_2 y_1, x_1 y_2\}, \max\{x_1 y_1, x_2 y_2\}) & \\ (x_1 \leq 0 \leq x_2, y_1 \leq 0 \leq y_2) & \\ (x_2 y_1, x_1 y_1) & (x_1 \leq 0 \leq x_2, y_2 \leq 0) \\ (x_1 y_2, x_2 y_1) & (x_2 \leq 0, 0 \leq y_1) \\ (x_1 y_2, x_1 y_1) & (x_2 \leq 0, y_1 \leq 0 \leq y_2) \\ (x_2 y_2, x_1 y_1) & (x_2 \leq 0, y_2 \leq 0) \end{cases}$$

By the above parametric representation of fuzzy numbers we get the following theorem concerning properties of end-points.

**Theorem 1** Denote  $x = (x_1, x_2) \in \mathcal{F}_b^{st}$ , where  $x_1, x_2 : I \rightarrow \mathbb{R}$ . Then the following properties (i)-(iii) hold:

(i)  $x_i \in C(I), i = 1, 2$ . Here  $C(I)$  is the set of all the continuous functions on  $I$ ;

(ii) There exists a unique  $m \in \mathbb{R}$  such that  $x_1(1) = x_2(1) = m$  and  $x_1(\alpha) \leq m \leq x_2(\alpha)$  for  $\alpha \in I$ ;

(iii) One of the following statements (a) and (b)

(a) Functions  $x_1, x_2$  are non-decreasing, non-increasing on  $I$ , respectively, with  $x_1(\alpha) < x_2(\alpha)$  for  $0 \leq \alpha < 1$ ;

(b)  $x_1(\alpha) = x_2(\alpha) = m$  for  $0 < \alpha \leq 1$ .

Conversely, under the above conditions (i) -(iii), if we denote

$$\mu_x(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leq \xi \leq x_2(\alpha)\} \quad (1.1)$$

then  $\mu_x$  is the membership function of  $x$ , i.e.,  $\mu_x \in \mathcal{F}_b^{st}$ .

Let a metric between  $x = (x_1(\cdot), x_2(\cdot)), y = (y_1(\cdot), y_2(\cdot))$  be defined as follows.

$$d(x, y) = \sup_{\alpha \in I} \sqrt{|x_1(\alpha) - y_1(\alpha)|^2 + |x_2(\alpha) - y_2(\alpha)|^2}$$

Then we get following result immediately.

**Theorem 2**  $(\mathcal{F}_b^{st}, d)$  is complete metric space.

## 2 Set of $L$ -fuzzy numbers

Denote a shape function by  $L : \mathbf{R} \rightarrow I$ , where  $L$  is upper semi-continuous and satisfies the following properties (i) - (iv):

- (i)  $L(0) = \max_{\mathbf{R}} L(\xi) = 1$ ; (ii)  $L(\xi)$  is strictly decreasing in  $\xi \geq 0$ ;
- (iii)  $L(-\xi) = L(\xi)$  for  $\xi \geq 0$ ; (iv)  $\sup\{\xi \in \mathbf{R} : L(\xi) > 0\} = 1$ .

In what follows we consider a set of  $L$ -fuzzy numbers  $\mathcal{F}_L = \{\mu \in \mathcal{F}_b^{st} : (a) \text{ or } (b) \text{ hold.}\}$  Let  $m \in \mathbf{R}$ ,  $\ell \geq 0$ . There exist two typical types (a) and (b) of  $\mathcal{F}_L$ .

$$(a) \ell > 0 \text{ and } \mu(\xi) = \begin{cases} L(\frac{m-\xi}{\ell}) & \text{for } \xi \leq m \\ L(\frac{\xi-m}{\ell}) & \text{for } \xi > m \end{cases}$$

$$(b) \ell = 0 \text{ and } \mu(\xi) = \begin{cases} 1 & \text{for } \xi = m \\ 0 & \text{for } \xi \neq m \end{cases}$$

In this section we introduce a total order relation  $\lambda$ -fuzzy max order  $\preceq_\lambda$  over  $\mathcal{F}_L$ . Here  $0 \leq \lambda \leq 1$  is given by decision makers. Let  $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_L$  with the center  $x_1(1)$ , the spread  $\ell_x = x_1(1) - x_1(0) \geq 0$  and center  $y_1(1)$ , spread  $\ell_y = y_1(1) - y_1(0) \geq 0$ . We define that  $x \preceq_\lambda y$  if and only if the following statements (i)-(iii) ([1]):

- (i)  $|\ell_y - \ell_x| \leq y_1(1) - x_1(1)$  for  $y_1(1) \geq x_1(1)$ ;
- (ii)  $\lambda|\ell_y - \ell_x| \leq y_1(1) - x_1(1) < |\ell_y - \ell_x|$  for  $y_1(1) > x_1(1)$  and  $\ell_y \neq \ell_x$ ;
- (iii)  $|y_1(1) - x_1(1)| < \lambda(\ell_y - \ell_x)$  for  $\ell_y - \ell_x > 0$ .

Furukawa [1] gives the following theorem so that two any  $L$ -fuzzy numbers can be compared to each other.

**Theorem F1** For any  $x, y \in \mathcal{F}_L$  it follows that one of the relations  $x \preceq_\lambda y$ , and  $y \preceq_\lambda x$  hold. Thus  $\preceq_\lambda$  is a total order relation over  $\mathcal{F}_L$ .

The following theorem plays an important role in comparing two  $L$ -fuzzy numbers.

**Theorem F2** For  $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_L$  satisfying  $\ell_x = x_1(1) - x_1(0) \geq 0$ ,  $\ell_y = y_1(1) - y_1(0) \geq 0$ , it follows that  $x \preceq_\lambda y$  means that (i) or (ii) hold.

- (i)  $\lambda\ell_x + x_1(1) < \lambda\ell_y + y_1(1)$  for  $\ell_y > \ell_x$ ;
- (ii)  $\lambda\ell_x + x_1(1) \leq \lambda\ell_y + y_1(1)$  for  $\ell_y \leq \ell_x$ .

## 3 Fuzzy optimization problems

Let  $0 \leq \lambda \leq 1$ . In this section we show criteria concerning the following optimization problem

$$\text{minimize } f(z) \text{ subject to } z \in \mathcal{C}_\lambda^\delta. \quad (P_\lambda^\delta)$$

where  $\mathcal{C}_\lambda^\delta$  is a feasible set in  $\mathcal{F}_L^n$  or  $\mathcal{F}_L^\infty$  and  $f : \mathcal{C}_\lambda^\delta \rightarrow \mathcal{F}_L$  is an objective function. We denote  $z \in \mathcal{F}_L^\infty$  by

$$z = (z_1, z_2, z_3, \dots) \text{ where } z_i \in \mathcal{F}_L \text{ for } i = 1, 2, \dots.$$

In what follows we consider

$$\mathcal{C}_\lambda^\delta = \{z \in \mathcal{F}_L^\infty : g_j(z) \preceq_\lambda (0, \delta_j)_L, j = 1, 2, \dots\}.$$

where  $g_j : \mathcal{C}_\lambda^\delta \rightarrow \mathcal{F}_L$ ,  $(0, \delta_j)_L \in \mathcal{F}_L$ , and  $\delta_j > 0$  are constants for  $j = 1, 2, \dots$ . Let  $\delta = (\delta_1, \delta_2, \dots) \in \mathbf{R}^\infty$ . If  $z^* \in \mathcal{C}_\lambda^\delta$  satisfies  $f(z^*) = \min\{f(z) : z \in \mathcal{C}_\lambda^\delta\}$  then  $z^*$  is called an optimal solution of  $(P_\lambda^\delta)$ .

In order to analyze  $(P_\lambda^\delta)$  which is general case, we may consider an  $\mathbf{R}$ -valued optimization problem

$$\text{minimize } f(z) \text{ subject to } z \in \mathcal{C}_\lambda^\delta \cap \mathbf{R}. \quad (P_0^\delta)$$

Then, letting  $f_0^* = \min_{z \in C_\lambda^\delta \cap \mathbf{R}} f(z)$ , we get  $f_0^* \in \mathbf{R}$  which gives the optimal value in  $\mathbf{R}$  as follows.

**Corollary 1** *Let  $f^* \in \mathbf{R}$ . Then there exists no  $L$ -fuzzy number  $f \in \mathcal{F}_L \setminus \mathbf{R}$  such that  $f = f^*$ , i.e.,  $f \preceq_\lambda f^*$  and  $f^* \preceq_\lambda f$ .*

From the above corollary we get the following lemma immediately.

**Lemma 1** *Denote  $f_1^\delta = \min\{f(z) : z \in C_\lambda^\delta\}$ ,  $f_2^\delta = \min\{f(z^c) : z \in C_\lambda^\delta\} \in \mathbf{R}$ ,  $f_3^\delta = \min\{f(z)^c : z \in C_\lambda^\delta\} \in \mathbf{R}$ ,  $f_0^* = \min\{f(z) : z \in C_0^0\} \in \mathbf{R}$ , where  $z^c \in \mathbf{R}$ ,  $f(z)^c \in \mathbf{R}$  are centers of  $z, f(z)$ , respectively, and  $C_0^0 = C_\lambda^\delta \cap \mathbf{R}$ .*

*If there exist  $f_i^\delta, i = 1, 2, 3$  and  $f_0^*$ , then it follows that  $f_1^\delta \in \mathbf{R}$  and that*

$$f_1^\delta = f_2^\delta = f_3^\delta \leq f_0^*.$$

*If  $\delta = 0$ , then  $f_1^0 = f_2^0 = f_3^0 = f_0^*$ .*

**Remark 1** *It follows that*

$$C_0^0 = \{z \in \mathbf{R}^\infty : g_j(z) \leq 0, j = 1, 2, \dots\} = C_\lambda^\delta \cap \mathbf{R}.$$

*When  $\delta_j > 0$  for some integer  $j$ , there exists an example such that  $f_1^\delta = f_2^\delta = f_3^\delta < f_4^\delta$ . See Example 1 below.*

In case that there exists an optimal solution of  $L$ -fuzzy optimization problems, by the above lemma, the solution means a real number.

**Theorem 3** *Denote  $f^* = \min\{f(z) : z \in C_\lambda^\delta\}$  and  $f_0^* = \min\{f(z) : z \in C_\lambda^\delta \cap \mathbf{R}\}$ . Suppose that  $(P_\lambda^\delta)$  has at least one optimal solution in  $C_\lambda^\delta$ . Then there exist  $f^*, f_0^*$  in  $\mathbf{R}$ , which satisfy  $f^* = f_0^*$ .*

## $L$ -fuzzied numbers

In what follows we introduce an idea of  $L$ -fuzzied numbers generalized by  $\mathcal{F}_b^{st}$ . Let  $x \in \mathcal{F}_L$ . The quadratic  $x^2$  of an  $L$ -fuzzy number  $x$  isn't necessarily  $L$ -fuzzy number but fuzzy number in  $\mathcal{F}_b^{st}$  (see [6]). For  $x = (x_1, x_2) \in \mathcal{F}_L$  and  $\alpha \in I$ , we have the following three cases:

- $x^2 = (x_1^2, x_2^2)$  if  $x_1(\alpha) \geq 0$ ;
- $x^2 = (x_1 x_2, \max[x_1^2, x_2^2])$  if  $x_1(\alpha) \leq 0 \leq x_2(\alpha)$ ;
- $x^2 = (x_2^2, x_1^2)$  if  $x_2(\alpha) \leq 0$ .

In this study we consider the left portion of the membership function  $\mu_{x^2}$  is more significant than the right portion of  $\mu_{x^2}$ . Denote an operator  $(\cdot)_L : \mathcal{F}_b^{st} \rightarrow \mathcal{F}_L$  such that  $(x)_L = (x_1(1), x_1(1) - x_1(0))_L$  for  $x = (x_1, x_2) \in \mathcal{F}_b^{st}$ . We call that  $(x)_L$  is an  $L$ -fuzzied number. Here the membership function of  $x$  is  $\mu_x(\xi) = L(\frac{x_1(1) - \xi}{x_1(1) - x_1(0)})_+$  for  $\xi \in \mathbf{R}$ ,  $L : \mathbf{R} \rightarrow \mathbf{R}_+$  is a shape function and  $\xi_+ = \max(\xi, 0)$  if  $\xi \in \mathbf{R}$ . For  $x \in \mathcal{F}_L$  we get the  $L$ -fuzzied number

$$(x^2)_L = (x_1(1)^2, x_1(1)^2 - x_i(0)x_j(0))_L,$$

where  $i = 1, j = 2$  if  $x_1(0)x_2(0) \leq 0$ ,  $i = j = 1$  if  $x_1(0)x_2(0) > 0$  and  $|x_1(0)| < |x_2(0)|$ ,  $i = j = 2$  if  $x_1(0)x_2(0) > 0$  and  $|x_1(0)| \geq |x_2(0)|$ .

Let a shape function be  $L(\xi) = (1 - |\xi|)_+$ . For an  $L$ -fuzzy number  $x = (\xi_0, \ell)_L$  with  $|\xi_0| \leq \ell$ , which has the membership function  $\mu_x(\xi) = L(\frac{\xi_0 - \xi}{\ell})_+$  for  $\xi \in \mathbf{R}$ . Then we get the membership function

$$\mu_{x^2}(\xi) = \begin{cases} \left(1 - \frac{\sqrt{\xi_0^2 - \xi}}{\ell}\right)_+ & \text{for } \xi < \xi_0^2; \\ \left(1 - \frac{\xi_0 - \sqrt{\xi}}{\ell}\right)_+ & \text{for } \xi \geq \xi_0^2. \end{cases}$$

In this case we construct an  $L$ -fuzzy numbers  $(x^2)_L$  with the same portion as the left one of

$\mu_{x^2}$ . It follows that  $(x^2)_L = (\xi_0^2, \ell^2)_L$ . For  $x \in \mathcal{F}_L$  and  $k \in \mathbf{R}$  we have  $(kx)_L = kx$ .

In the following example we consider  $L$ -fuzzy optimization problem with a fuzzy objective function and fuzzy constraints.

**Example 1** Let  $z = (u, v) \in \mathcal{F}_L^2$  and  $\lambda \in I$ . Fuzzy functions  $F, g_j, j = 1, 2, 3$ , are as follows  $(P_\lambda^\delta)$ :

$$\begin{aligned} F(z) &= -u - v; \\ g_1(z) &= -u \preceq_\lambda (0, \delta_1)_L; \\ g_2(z) &= -v \preceq_\lambda (0, \delta_2)_L; \\ g_3(z) &= (u^2)_L + (v^2)_L \preceq_\lambda (1, \delta_3)_L. \end{aligned}$$

Here  $(0, \delta_1)_L, (0, \delta_2)_L, (1, \delta_3)_L$  are  $L$ -fuzzy numbers and  $(u^2)_L = (u_1(1)^2, \ell_{u^2})_L, (v^2)_L = (v_1(1)^2, \ell_{v^2})_L$  are  $L$ -fuzzified numbers.

The minimum of the above problem is attained at  $u_1(1) = v_1(1) = (-\sqrt{\frac{1+\lambda\delta_3}{2}}, 0)_L$ , which means that  $\min_z f(z) = (-\sqrt{\frac{1+\lambda\delta_3}{2}}, 0)_L$  and  $u^* = v^* = (-\sqrt{\frac{1+\lambda\delta_3}{2}}, 0)_L$ . When  $\lambda = 0$  and  $\delta_j = 0, j = 1, 2, 3$ , then the real type of optimization problem  $(P_0^0)$  gives  $-\sqrt{2} \leq f(z) \leq 0$  in  $\mathbf{R}$  and  $u^* = v^* = 1/\sqrt{2} \in \mathbf{R}$ .

This example shows that there exists a unique optimal solution of  $L$ -fuzzy number of fuzzy optimization problem  $(P_\lambda^\delta)$  with a fuzzy coefficient, where  $(P_\lambda^\delta)$  is an optimization problem with  $\mathbf{R}$ -valued coefficients if  $\ell_z = 0$  and  $(P_\lambda^\delta)$  is fuzzy type if  $\ell_z \neq 0$ , where  $\ell_z$  is the spread of  $z \in \mathcal{C}_\lambda^\delta$ . Therefore the optimal solution to the real type  $(P_0^0)$  is the same as solution to the fuzzy type  $(P_\lambda^\delta)$  concerning  $\lambda = 0$  and  $\ell_z = 0$ .

## 4 Oil well equations with $\mathbf{R}$ -valued functions

In [8] they discuss exponential decay problems, e.g., machine replacement and oil well extraction, etc. They analyze optimization problems for each oil well to determine its optimal replacement schedule. In order to give a mathematical model we introduce the following notations.

- $C(t)$ : the quality remaining in the well at time  $t$
- $D > 0$ : rate of oil extraction
- $P$ : unit profit of oil (sufficiently large)

As the oil reserves get depleted, the rate of extraction eventually decreases to uneconomic levels, making it worthwhile to abandon the well and drill a new one at a cost  $f(\nu)$ . Here  $\nu$  is capacity of the well,  $f$  is continuously differentiable function. Assume that  $V \in \mathbf{R}$  and that  $0 \leq \nu \leq V, f(0) = 0, f'(\nu) \geq 0$ .

Then they get the following rate of oil extraction  $C'(t) = -DC(t)$  with  $C(0) = \nu$ . Then  $C(t) = \nu e^{-Dt}$ .

Moreover they discuss deterministic discounted models in case of horizon models with a continuous discount rate  $r > 0$ . Let  $T = \{t_i : i = 1, 2, \dots\}$  be a sequence of drilling times such that  $0 \leq t_i < t_{i+1}$  and  $\mathcal{V} = \{\nu_i : i = 1, 2, \dots\}$  a sequence of corresponding oil well capacity such that  $0 \leq \nu_i \leq V$ . They get a value of the net profit function

$$\begin{aligned} & J(T, \mathcal{V}) \\ &= \sum_{i=1}^{\infty} \left[ \int_{t_i}^{t_{i+1}} P \nu_i e^{-D(t-t_i)} e^{-r(t-t_i)} dt - f(\nu_i) \right] e^{-rt_i}. \end{aligned}$$

They consider maximizing problems of  $J(T, \mathcal{V})$  and show the following results.

**Theorem STU** The following statements 1) and 2) hold:

- 1) There exist optimal sequences  $T^* = \{t_i^* : i = 1, 2, \dots\}$  and  $\mathcal{V}^* = \{\nu_i^* : i = 1, 2, \dots\}$  such that

$$\max_{T, \mathcal{V}} J(T, \mathcal{V}) = J(T^*, \mathcal{V}^*).$$

- 2) It follows that  $t_1^* = 0$ ,  $t_2^* = t_{i+1}^* - t_i^*$ ,  $\nu_1^* = \nu_i^*$  for  $i = 1, 2, \dots$ .

Let  $T = t_2^*$ ,  $\nu = \nu_1^*$ . Then

$$J = J(T^*, \mathcal{V}^*) = \int_0^T P \nu e^{-(D+r)t} dt - f(\nu) + J e^{-rT},$$

$$\text{or } J = \frac{\nu}{1 - e^{-rT}} \left[ \frac{P(1 - e^{-(D+r)T})}{D+r} - \frac{f(\nu)}{\nu} \right].$$

In [8] they assume that there exists an optimal solution to the problem of maximizing  $J$ . They mention that a sufficiently large value of  $P$  will guarantee some drilling optimal but that the question of the existence is beyond the scope of the paper.

In the following section we show the maxi-max theorem [4] and we show an extension. Moreover we apply the extension theorem to the existence discussion for optimal solutions of  $J$ .

## 5 Maxi-max theorems

In [4] the author get the following theorem concerning the existence of dynamical programming.

**Theorem I[IWAMOTO]** Let  $X$  be a set and  $g : X \times \mathbf{R} \rightarrow \mathbf{R}$  such that  $g(x, \cdot) : \mathbf{R} \rightarrow \mathbf{R}$  is non-decreasing for each  $x \in X$ . Denote a set-valued function by  $\mathcal{Y}(\cdot) : X \rightarrow 2^{\mathbf{R}}$ , where  $2^{\mathbf{R}}$  is the power

set of  $\mathbf{R}$  and a graph by  $G(\mathcal{Y}) = \{(x, y) \in X \times \mathbf{R} : x \in X, y \in \mathcal{Y}(x)\}$ .

If a function  $h : G(\mathcal{Y}) \rightarrow \mathbf{R}$  satisfy

$$\exists \max_{x \in X} g(x, \max_{y \in \mathcal{Y}(x)} h(x, y)) = c$$

then there exists  $\max_{x \in X, y \in \mathcal{Y}(x)} g(x, h(x, y))$  such that  $c = \max_{x \in X, y \in \mathcal{Y}(x)} g(x, h(x, y))$ .

In order to guarantee the existence of optimal solutions of the maximizing problems  $J$  we discuss the following extension of the above theorem.

**Theorem 4** Let  $X$  and  $Y$  are sets. Denote  $g : X \times Y \rightarrow \mathbf{R}$  such that  $g(x, \cdot) : Y \rightarrow \mathbf{R}$  satisfies

$$\exists g(x, \max_{y \in \mathcal{Y}(x)} h(x, y)) \text{ and that}$$

$g(x, h(x, y)) \leq g(x, \max_{y \in \mathcal{Y}(x)} h(x, y))$  for  $x \in X$ . Denote a set-valued function by  $\mathcal{Y}(\cdot) : X \rightarrow 2^{\mathbf{R}}$ .

Here  $2^{\mathbf{R}}$  is the power set of  $\mathbf{R}$  and a graph by  $G(\mathcal{Y}) = \{(x, y) \in X \times Y : x \in X, y \in \mathcal{Y}(x)\}$ . Let a function  $h : G(\mathcal{Y}) \rightarrow \mathbf{R}$  satisfy

$$\exists \max_{x \in X} g(x, \max_{y \in \mathcal{Y}(x)} h(x, y)) = c. \text{ Then we get}$$

$$\exists \max_{x \in X, y \in \mathcal{Y}(x)} g(x, h(x, y)) \text{ such that } c = \max_{x \in X, y \in \mathcal{Y}(x)} g(x, h(x, y)).$$

Let  $\mathcal{J}(T, \mathcal{V}) = g(T, \mathcal{V}) = \sum_{i=1}^{\infty} h(T, \mathcal{V})$ , and

$$h(T, \mathcal{V}) = \left[ \frac{P(1 - e^{-(D+r)(t_{i+1} - t_i)}) \nu_i}{D+r} - f(\nu_i) \right] e^{-rt_i}.$$

Denote  $X = \{T = (t_1, t_2, \dots)\}$ , and  $Y = \{\mathcal{V} = (\nu_1, \nu_2, \dots)\}$ . From the above we get

$$\max_{\mathcal{V} \in Y} h(T, \mathcal{V}^*) = \left[ \frac{P(1 - e^{-(D+r)(t_{i+1} - t_i)}) \nu_i^*}{D+r} - f(\nu_i^*) \right] e^{-rt_i}$$

Here  $\mathcal{V}^* = \{\nu_i^*\}$ ,  $\nu_i^* = \min(V, \bar{\nu}_i)$  and  $f'(\bar{\nu}_i) = \frac{P(1 - e^{-(D+r)(t_{i+1} - t_i)})}{D+r}$ .

**Assumption.**  $t_1^* = 0$  and  $\exists T > 0$  such that  $\Delta t_i \leq T$ .

Denote  $T^* = \{t_i^*\}$  such that  $T = t_{i+1}^* - t_i^* = t_2^*$ .

$\nu_i^* = \min(V, \bar{\nu}_i)$  and  $f'(\bar{\nu}_i) = \frac{P(1 - e^{-(D+r)T})}{D+r}$  so  $\nu_i^* = \nu_{i+1}^*$  for  $i \geq 1$ .

Then, by the extension of the maxi-max theorem, we have

$$\begin{aligned}\max_{T \in X} g(T, \max_{V \in Y} V) &= g(T^*, V^*) \\ &= \max_{T \in X, V \in Y} g(T, V).\end{aligned}$$

## 6 Fuzzy functions

Consider a function  $x(t) : \mathbf{R} \rightarrow \mathcal{F}_b^{st}$ . Then  $x(t)$  is said to be a fuzzy function. In [5] we find the following definition of fuzzy functions.

$$\begin{aligned}x(t, \cdot) &= \{(x_1(t, \alpha), x_2(t, \alpha))^T \in \mathbf{R}^2 : \alpha \in I\} \\ &= (x_1(t, \cdot), x_2(t, \cdot))\end{aligned}$$

for  $t \in [t_1, t_2]$ . Denote  $x(t) = (x_1(t), x_2(t))$ .

An  $L$ -fuzzy function  $x(t) = (x_1(t), x_2(t)) : \mathbf{R} \rightarrow \mathcal{F}_b^{st}$  is  $H$ -differentiable at  $t$  in the sense of Hukuhara if there exists an  $\eta \in \mathcal{F}_b^{st}$  such that (i) and (ii) hold as  $h \rightarrow +0$ .

(i)  $x(t+h) = x(t) + h\eta + o(h)$ ; (ii)  $x(t) = x(t-h) + h\eta + o(h)$ .

Here  $o(h) = (o_1(h), o_2(h)) \in C[0, \varepsilon] \times C[0, \varepsilon]$  with  $\varepsilon > 0$ , which means that

$$\lim_{|h| \rightarrow 0} \frac{d(o(h), 0)}{|h|} = 0.$$

Then  $x(t) = (x_1(t), x_2(t))$  is  $H$ -differentiable at  $t$  if and only if  $x_1(t, \alpha), x_2(t, \alpha)$  are differentiable in  $t$  for each  $\alpha \in I$  such that  $\eta = (\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}) \in \mathcal{F}_b^{st}$ .

In [5] the author discuss the integration of fuzzy function  $x(t)$ .

**Definition 2** An  $L$ -fuzzy function

$x(t, \cdot) = (x_1(t, \cdot), x_2(t, \cdot))$  is called integrable over  $[t_1, t_2]$  if  $x_1(t, \alpha)$  and  $x_2(t, \alpha)$  are integrable over  $[t_1, t_2]$  for  $\alpha \in I$ . Define

$$\int_{t_1}^{t_2} x(s, \cdot) ds$$

$$= \{(\int_{t_1}^{t_2} x_1(s, \alpha) ds, \int_{t_1}^{t_2} x_2(s, \alpha) ds)^T \in \mathbf{R}^2 : \alpha \in I\}.$$

## 7 Oil well equations with $L$ -fuzzy functions

In the same way as analyzing oil well equations of  $\mathbf{R}$ -valued functions we consider the following problem with  $0 \leq \lambda \leq 1$ . We consider the rate of oil extraction  $D_L$  as a constant  $L$ -fuzzy number  $D_L = (D_1, D_2) \in \mathcal{F}_L$  such that  $0 \preceq_\lambda D_L$ , where  $D_1(\alpha)$  is the left end-point of the  $\alpha$ -cut set and  $D_1(\alpha) > 0$  for  $\alpha \in I$ . Then we assume that the oil quality and unit profit of oil are  $L$ -fuzzy function and  $L$ -fuzzy number, respectively.

- $C_L(t) = (C_1(t), C_2(t)) \in \mathcal{F}_L$  :  $L$ -fuzzy function which means the quality remaining in the well at time  $t$
- $P_L = (P_1, P_2) \in \mathcal{F}_L$  : unit profit of oil with  $0 \preceq_\lambda P_L$

In this section we consider the following notations.  $\nu \in \mathbf{R}$  is a capacity of the well,  $f(\nu)$  is renewal cost which is continuously differentiable with  $f(0) = 0$  and  $f'(\nu) \geq 0$  and  $V$  is the upper bound such that  $0 \leq \nu \leq V$ .

Then we get an initial value problem of  $L$ -fuzzy differential equation  $\frac{dC_L}{dt}(t) + D_L C_L(t) = 0$  with  $C_L(0) = \nu$ , where  $0 \in \mathbf{R}$ . It follows that as long as  $C_1(t) \geq 0$

$$C_1'(t) + D_1 C_1(t) = 0$$

$$C_2'(t) + D_2 C_2(t) = 0$$

with  $C_1(0) = C_2(0) = \nu$ . Therefore

$$C_1(t, \alpha) = \nu e^{-D_1(\alpha)t}, C_2(t, \alpha) = \nu e^{-D_2(\alpha)t}$$



for  $t \geq 0, \alpha \in I$ . We can solve some cases of the above fuzzy differential equations (see [6]). Without using the above information about the  $L$ -fuzzy function  $C_L(t)$  we can find optimal solutions for maximizing problems of values of the net profit function with  $L$ -fuzzy value concerning  $T = \{t_i \in \mathbf{R} : i = 1, 2, \dots\}$  such that  $0 \leq t_i < t_{i+1}$  and  $\mathcal{V} = \{\nu_i \in \mathbf{R} : i = 1, 2, \dots\}$  such that  $0 \leq \nu \leq V$  as follows.

$$\mathcal{J}(T, \mathcal{V}) = \sum_{i=1}^{\infty} \left[ \int_{t_i}^{t_{i+1}} P_L \nu_i C_L(t - t_i) e^{-r(t-t_i)} dt - f(\nu_i) \right] e^{-rt_i}.$$

It can be seen that  $\mathcal{J}(T, \mathcal{V}) = (J_1, J_2) \in \mathcal{F}_L$ , where

$$\begin{aligned} J_1 &= \sum_{i=1}^{\infty} \left[ \int_{t_i}^{t_{i+1}} P_1 \nu_i e^{-D_1 t} e^{-r(t-t_i)} dt - f(\nu_i) \right] e^{-rt_i}, \\ J_2 &= \sum_{i=1}^{\infty} \left[ \int_{t_i}^{t_{i+1}} P_2 \nu_i e^{-D_2 t} e^{-r(t-t_i)} dt - f(\nu_i) \right] e^{-rt_i}. \end{aligned}$$

Here  $J_1(T, \mathcal{V}, \alpha), J_2(T, \mathcal{V}, \alpha)$  are  $\mathbf{R}$ -valued functions defined on  $X \times Y \times I$ . Denote

$$\begin{aligned} X &= \{T = (t_1, t_2, \dots) : t_i \in \mathbf{R}, 0 \leq t_i < t_{i+1}\}, \\ Y &= \{\mathcal{V} = (\nu_1, \nu_2, \dots) : \nu_i \in \mathbf{R}, 0 \leq \nu \leq V\}. \end{aligned}$$

Consider the maximizing problem of  $\mathcal{J}(T, \mathcal{V})$ . We get the following theorem by applying Theorem 3 and STU.

**Theorem 5** *The following statements 1) and 2) hold.*

- 1) *There exists at least one pair of optimal sequences  $T^* = \{t_i^* : i = 1, 2, \dots\}$  and  $\mathcal{V}^* = \{\nu_i : i = 1, 2, \dots\}$  such that*

$$\max_{T, \mathcal{V} \in X \times Y} \mathcal{J}(T, \mathcal{V}) = \mathcal{J}(T^*, \mathcal{V}^*) \in \mathbf{R}.$$

- 2) *It follows that  $t_1^* = 0, t_2^* = t_{i+1}^* - t_i^*, \nu_1^* = \nu_i^* \in \mathbf{R}$  for  $i = 1, 2, \dots$ . Let  $T = t_2^*, \nu = \nu_1^*$ .*

*Then*

$$\mathcal{J}(T^*, \mathcal{V}^*) = \frac{\nu}{1 - e^{-rT}} \left[ \frac{P(1 - e^{-(D+r)T})}{D + r} - \frac{f(\nu)}{\nu} \right]$$

*Here  $P, D \in \mathbf{R}$  are respective centers of  $L$ -fuzzy numbers  $P_L, D_L$ .*

## References

- [1] N. FURUKAWA (1999), *Mathematical Methods of Fuzzy Optimization* (in Japanese), Morikita Pub., Tokyo.
- [2] Jr. R. GOETSCHTEL, W. VOXMAN (1983), "Topological Properties of Fuzzy Numbers," *Fuzzy Sets and Systems*, vol 9, 87-99.
- [3] Jr. R. GOETSCHTEL, W. VOXMAN (1986), "Elementary Fuzzy Calculus," *Fuzzy Sets and Systems*, vol 18, 31-43.
- [4] S. IWAMOTO (1985), "Sequential Minimaximization under Dynamic Programming Structure," *J. Math. Appl.*, vol 108, 267-282.
- [5] O. KALEVA (1990), "The Cauchy Problem for Fuzzy Differential Equation," *Fuzzy Sets and Systems*, vol 35, 389-396.
- [6] S. SAITO (in press), "On Some Topics of Fuzzy Differential Equations and Fuzzy Optimization Problems via a Parametric Representation of Fuzzy Numbers," *Differential Equations and Applications, Vol. 3*, Nova Science Publishers, Inc., New York.
- [7] S. SAITO, "On Boundary Value Problems of Fuzzy Differential Equations," *Proceedings*

*of the Second Vietnam-Japan Symposium on  
Fuzzy Systems and Applications*, ed. N. H.  
PHOUNG and K. YAMADA.

S. P. SETHI, G. L. THOMPSON and V.  
UDAYABHANU (1985), "Profit Maximiza-  
tion Models for Exponential Decay Pro-  
cesses," *European J. OR* , vol 22, 101-115.